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Multiple hypergeometric functions and 9-j coefficients

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Abstract. It is well known that the 9-j recoupling coefficient appearing in the quantum theory of angular momentum has 72 symmetries. However, the triple-sum series expression for the 9-j coefficient exhibits none of these symmetries. Here a *stretched* 9-j coefficient, for which a closed-form (single-term) expression exists, is considered and the type of summation theorems the triple-sum series reduces is investigated for any of the 72 symmetries. Apart from well known single-summation theorems for hypergeometric functions, this analysis gives rise to new summation theorems for double and triple hypergeometric functions.

1. Introduction

The basic idea of this paper was described in Srinivasa Rao and Van der Jeugt (1994, here referred to as I). In that paper, the doubly stretched 9-j coefficient

$$\left\{ \begin{array}{ccc} a & b & c \\ d & e & f \\ a+d & a+d+i & i \end{array} \right\} \tag{1}$$

was considered. This coefficient has a closed-form expression (Sharp 1967). On the other hand, 9-j coefficients can also be determined by means of the triple-sum series of Jucys and Bandzaitis (1977). This triple-sum series can be written down for (1) directly or for any of the 71 remaining symmetries (Jahn and Hope 1954) of (1). Due to the inherent asymmetry of the triple-sum series, all these 72 expressions would appear to be different. In I, it was observed that for certain symmetries of (1), the triple sum would reduce to a single sum and stating that these single sums are equal to the closed-form expression for (1) thus leads to well known summation theorems for generalized hypergeometric functions. Moreover, it was pointed out that apart from yielding manifestations of well known summation theorems, a complete study of all 72 symmetries opens the prospect of finding genuinely new summation theorems. In the present paper, we present such a complete analysis and indeed some summation theorems for double and triple hypergeometric functions arise. We have not found these in the literature, and hence we believe that they are new results.

In section 2, we recall the notation for generalized hypergeometric functions and define multiple hypergeometric functions. In section 3, the triple-sum series is considered for the

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72 symmetries of the coefficient (1). It turns out that of the 72 cases, four yield a single term (or a closed-form expression); 16 yield a single sum of three different types; 20 yield a double sum of six different types; and 32 yield a triple sum of four different types. All the single sums are manifestations of well known summation theorems. Two of the six double summations and two of the four triple summations cannot be summed using known results, and they give rise to new summation theorems for special double and triple hypergeometric functions, while the rest can be summed using known single summation theorems. These summation formulae and some further specializations are discussed in section 4, followed by some conclusions.

2. Generalized and multiple hypergeometric functions

The Gauss function or hypergeometric function, nowadays usually represented by the symbol ${}_2F_1[{a,b \atop c}; z]$, is defined as

$$ {}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix}; x\right] = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!}. \tag{2}$$

Herein, a, b and c are the (complex) parameters, x is the (complex) variable of the function, and the common notation for the Pochammer symbol has been used:

$$ (a)_n = a(a+1)(a+2)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)} \qquad (a)_0 = 1 \tag{3}$$

where Γ is the classical Γ-function. For a historical introduction to the Gauss function, and a survey of its properties, we refer the reader to the first monograph on hypergeometric series by Bailey (1935) and to Slater (1966). The idea of extending the number of parameters in the Gauss function occurred for the first time in the work of Clausen (1828), and these generalized hypergeometric functions were studied by Saalschütz (1890), Dixon (1903) and Dougall (1907). Much of the theory was summarized and extended by Bailey (1935; see also Slater 1963). The standard notation and definition for a generalized hypergeometric function is

$$ {}_AF_B\left[\begin{matrix} a_1, \ldots, a_A \\ b_1, \ldots, b_B \end{matrix}; x\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_A)_n}{(b_1)_n \cdots (b_B)_n} \frac{x^n}{n!}. \tag{4}$$

In a more compact notation, devised by Burchnall and Chaundy (1941) for multiple hypergeometric functions, the whole list of parameters $(a_1, \ldots, a_A)$ is not written explicitly but simply denoted by (a). Thus, (4) would be written as ${}_AF_B\left[{(a) \atop (b)}; x\right]$.

When one of the numerator parameters a_j is a negative integer, function (4) becomes a terminating series. Some of the most well known *summation theorems* for generalized hypergeometric functions are of this type; moreover, they are usually for unit argument $x = 1$. Vandermonde's theorem (which is the terminating form of the well known Gauss theorem, see Slater (1966)) reads as

$$ {}_2F_1\left[\begin{matrix} a, -m \\ c \end{matrix}; 1\right] = \frac{(c-a)_m}{(c)_m}. \tag{5}$$

In such summation theorems it is always understood that the termination is determined by $-m$ (hence $-a$ and $-c$ do not belong to $\{0, 1, \ldots, m-1\}$), although it is common not to mention this assumption explicitly. Another famous summation theorem is due to Saalschütz:

$$
{}_3F_2\left[\begin{matrix} a, b, -m \\ c, d \end{matrix}; 1\right] = \frac{(c-a)_m(c-b)_m}{(c)_m(c-a-b)_m} \tag{6}
$$

for $a+b-m+1 = c+d$. Some different summation theorems, which were discovered relatively recently, are due to Minton (1970) and Karlsson (1971) (see equation (1.9.3) of Gasper and Rahman (1990)). Minton's theorem reads as

$$
{}_{r+2}F_{r+1}\left[\begin{matrix} a, b, b_1+m_1, \ldots, b_r+m_r \\ b+1, b_1, \ldots, b_r \end{matrix}; 1\right] = \frac{\Gamma(b+1)\Gamma(1-a)}{\Gamma(1+b-a)} \frac{(b_1-b)_{m_1}\cdots(b_r-b)_{m_r}}{(b_1)_{m_1}\cdots(b_r)_{m_r}} \tag{7}
$$

where $m_1, \ldots, m_r$ are non-negative integers. Using this result, Karlsson deduced the following zero-balanced terminating series summation:

$$
\begin{aligned}
&{}_{r+1}F_r\left[\begin{matrix} -m_1-m_2-\cdots-m_r, b_1+m_1, \ldots, b_r+m_r \\ b_1, \ldots, b_r \end{matrix}; 1\right] \\
&\qquad = (-1)^{m_1+\cdots+m_r} \frac{(m_1+\cdots+m_r)!}{(b_1)_{m_1}\cdots(b_r)_{m_r}}.
\end{aligned} \tag{8}
$$

There are other summation theorems for generalized hypergeometric functions (see Slater 1966), but the only ones that appear in connection with this paper are (5)–(8).

For multiple hypergeometric functions that depend on more variables $x, y, \ldots$ the general theory is less advanced than for generalized hypergeometric functions with one variable. In addition the notation is not uniform and often confusing. Appell was the first author to study double hypergeometric functions systematically. The standard work on Appell series is Appell and Kampé de Fériet (1926). A general double hypergeometric function is known as the Kampé de Fériet function and is defined by (Kampé de Fériet 1921)

$$
F^{A:B}_{C:D}\left[\begin{matrix} (a) \\ (c) \end{matrix} : \begin{matrix} (b) \\ (d) \end{matrix}; \begin{matrix} (b') \\ (d') \end{matrix}; x, y\right] = \sum_{m,n=0}^{\infty} \frac{\prod_{j=1}^{A}(a_j)_{m+n}}{\prod_{j=1}^{C}(c_j)_{m+n}} \frac{\prod_{j=1}^{B}(b_j)_m(b'_j)_n}{\prod_{j=1}^{D}(d_j)_m(d'_j)_n} \frac{x^m y^n}{m!n!}. \tag{9}
$$

Herein, the parameters a_j and c_j appear with index $m+n$ in the Pochammer symbols. They are the coupling parameters and are responsible for the fact that (9) cannot be written as the product of two single hypergeometric functions in x and y separately. In (9), the number of parameters with index m is the same as that with index n. In general, this need not always be the case and therefore we define the following double hypergeometric function:

$$
F^{A:B;B'}_{C:D;D'}\left[\begin{matrix} (a) \\ (c) \end{matrix} : \begin{matrix} (b) \\ (d) \end{matrix}; \begin{matrix} (b') \\ (d') \end{matrix}; x, y\right] = \sum_{m,n=0}^{\infty} \frac{\prod_{j=1}^{A}(a_j)_{m+n}}{\prod_{j=1}^{C}(c_j)_{m+n}} \frac{\prod_{j=1}^{B}(b_j)_m}{\prod_{j=1}^{D}(d_j)_m} \frac{\prod_{j=1}^{B'}(b'_j)_n}{\prod_{j=1}^{D'}(d'_j)_n} \frac{x^m y^n}{m!n!}. \tag{10}
$$

This is, in fact, a special case of a very general function defined by Srivastava and Daoust (1969). For double hypergeometric functions there appears to be only one coupling, namely $m+n$. For triple hypergeometric functions with summation indices m, n and p, couplings of the form $m+n$, $m+p$, $n+p$ or $m+n+p$ can occur and hence the general notation becomes more complicated. Inspired by the general Srivastava and Daoust (1969) notation, we define here:

$$F^{A:B;B';B''}_{C:D;D';D''}\left[\begin{matrix}(a:\theta)\\(c:\psi)\end{matrix}:\begin{matrix}(b)\\(d)\end{matrix};\begin{matrix}(b')\\(d')\end{matrix};\begin{matrix}(b'')\\(d'')\end{matrix};x,y,z\right]$$
$$=\sum_{m,n,p=0}^{\infty}\frac{\prod_{j=1}^{A}(a_j)_{\theta_j^1 m+\theta_j^2 n+\theta_j^3 p}}{\prod_{j=1}^{C}(c_j)_{\psi_j^1 m+\psi_j^2 n+\psi_j^3 p}}\frac{\prod_{j=1}^{B}(b_j)_m}{\prod_{j=1}^{D}(d_j)_m}\frac{\prod_{j=1}^{B'}(b'_j)_n}{\prod_{j=1}^{D'}(d'_j)_n}\frac{\prod_{j=1}^{B''}(b''_j)_p}{\prod_{j=1}^{D''}(d''_j)_p}\frac{x^m y^n z^p}{m!n!p!} \tag{11}$$

where $(a:\theta)$ stands for $(a_1:\theta_1^1\theta_1^2\theta_1^3,\ldots,a_A:\theta_A^1\theta_A^2\theta_A^3)$ with $\theta_j^i\in\{0,1\}$, and similarly for $(c:\psi)$. In order to illustrate the above notation, we give one example:

$$F^{0:1;1;1}_{2:0,0;0}\left[\begin{matrix}\text{—}\\c_1:110,c_2:011\end{matrix}:\begin{matrix}b\\\text{—}\end{matrix};\begin{matrix}b'\\\text{—}\end{matrix};\begin{matrix}b''\\\text{—}\end{matrix};x,y,z\right]=\sum_{m,n,p=0}^{\infty}\frac{(b)_m(b')_n(b'')_p}{(c_1)_{m+n}(c_2)_{n+p}}\frac{x^m y^n z^p}{m!n!p!}. \tag{12}$$

For some of the existing literature on multiple hypergeometric functions, the reader is referred to Exton (1976). To our knowledge, the only summation theorems for multiple hypergeometric functions are for some very special generalized Kampé de Fériet functions (see Exton 1976 p 147). In this paper, we find some interesting summation theorems for double and triple hypergeometric functions. One of our results, for example, takes the following simple form in the notation of (9):

$$F^{0:3}_{1:1}\left[\begin{matrix}\text{—}\\\beta\end{matrix}:\begin{matrix}\alpha-\gamma,\beta+s,-r\\\alpha+s\end{matrix};\begin{matrix}\gamma-\alpha,\beta+r,-s\\\gamma+r\end{matrix};1,1\right]=\frac{(\alpha)_s(\gamma)_r}{(\gamma)_s(\alpha)_r} \tag{13}$$

where α,β,γ are complex numbers and $r,s\in\mathbb{N}$ determine the termination of the series.

3. The 9-j coefficient

The 9-j coefficient, or ls–jj transformation coefficient, plays an important role in the quantum theory of angular momentum (Wigner 1940, Biedenharn and Louck 1981). It is either given as a double sum over a product of six 3-j coefficients or as a single sum over a product of three 6-j coefficients. From the expression in terms of 3-j coefficients and the symmetrics of the 3-j coefficient, it can be established that the 9-j coefficient has 72 symmetries. Another expression for the 9-j coefficient is the triple-sum series of Jucys and Bandzaitis (1977). This expression has proved to be useful in numerical computations (Srinivasa Rao *et al* 1989, Srinivasa Rao and Rajeswari 1993), however, it does not exhibit any of the 72 symmetries. The Jucys–Bandzaitis triple sum is given by

$$\begin{Bmatrix}a&b&c\\d&e&f\\g&h&i\end{Bmatrix}=(-1)^{x_5}\frac{(d,a,g)(b,e,h)(i,g,h)}{(d,e,f)(b,a,c)(i,c,f)}$$
$$\times\sum_{x,y,z}\frac{(-1)^{x+y+z}}{x!y!z!}\frac{(x_1-x)!(x_2+x)!(x_3+x)!}{(x_4-x)!(x_5-x)!}\frac{(y_1+y)!(y_2+y)!}{(y_3+y)!(y_4-y)!(y_5-y)!}$$
$$\times\frac{(z_1-z)!(z_2+z)!}{(z_3-z)!(z_4-z)!(z_5-z)!}\frac{(p_1-y-z)!}{(p_2+x+y)!(p_3+x+z)!} \tag{14}$$

where

$$\begin{array}{lll}
x_1 = 2f & y_1 = -b+e+h & z_1 = 2a \\
x_2 = d+e-f & y_2 = g+h-i & z_2 = -a+b+c \\
x_3 = c-f+i & y_3 = 2h+1 & z_3 = a+d+g+1 \\
x_4 = -d+e+f & y_4 = b+e-h & z_4 = a+d-g \\
x_5 = c+f-i & y_5 = g-h+i & z_5 = a-b+c \\
p_1 = a+d-h+i & p_2 = -b+d-f+h & p_3 = -a+b-f+i
\end{array} \tag{15}$$

and

$$(a,b,c) = \left[\frac{(a-b+c)!(a+b-c)!(a+b+c+1)!}{(-a+b+c)!}\right]^{1/2}. \tag{16}$$

In (14), $a, b, \ldots, i$ are all integers or half-integers, and the three rows and three columns must form triads (i.e. (a, b, c) is a triad if $-a+b+c$, $a-b+c$ and $a+b-c$ are non-negative integers). Moreover, the summation indices x, y, z in (14) assume all integer values such that the factorials are non-negative. Explicitly, this means that

$$\begin{aligned}
&0 \leqslant x \leqslant \min(x_4, x_5) \\
&\max(0, -p_2 - x) \leqslant y \leqslant \min(y_4, y_5) \\
&\max(0, -p_3 - x) \leqslant z \leqslant \min(z_4, z_5, p_1 - y).
\end{aligned}$$

Sharp (1967) classified stretched 9-j coefficients (a triad (a, b, c) is stretched if one of the numbers $-a+b+c$, $a-b+c$, $a+b-c$ is zero). In particular, he derived a single-term expression for the following doubly stretched 9-j coefficient (see also Varshalovich *et al* 1975):

$$\left\{\begin{matrix} a & b & c \\ d & e & f \\ a+d & a+d+i & i \end{matrix}\right\}. \tag{17}$$

This particular 9-j coefficient was also the subject of I. If one uses the triple-sum expression (14) on the following particular symmetry of (17), the values of the numbers (15) are such that the triple sum reduces to a single term and one finds that

$$\begin{aligned}
\left\{\begin{matrix} a & b & c \\ d & e & f \\ a+d & a+d+i & i \end{matrix}\right\} &= \left\{\begin{matrix} a+d & a+d+i & i \\ a & b & c \\ d & e & f \end{matrix}\right\} \\
&= (-1)^{d-e+f} \frac{(a+d+i, b, e)}{(a,b,c)(d,e,f)(i,c,f)} \\
&\quad \times \left[\frac{(2a)!(2d)!(2i)!}{(2a+2d+1)(2a+2d+2i+1)!}\right]^{1/2}.
\end{aligned} \tag{18}$$

Thus, for a particular symmetry of (17), the triple-sum expression reduces to a single term. Due to the asymmetry of (14), the triple sum does not always reduce to a single term for the other symmetries of (17). This was the basic observation in I: consider any symmetry of (17) and use (14); this (usually) gives rise to a single-, double- or triple-sum series which has to be equal to the single term (18), hence a summation theorem follows. Several of the symmetries have already been considered in I. It was shown that, in some cases, the triple sum reduces to a single sum and that this single sum was a manifestation of the Vandermonde, Saalschütz or Karlsson theorem. An example where the triple sum reduces to a double sum and another where the triple sum remained a triple sum were given in I.

In this paper, we give a complete classification of the summations that appear for the 72 symmetries and study the consequent summation theorems. The results are easy to obtain, but require a careful analysis of the parameters (15) for each of the 72 symmetries of (17).

Before summarizing these results, it is convenient to introduce a shorthand notation for a symmetry of (17). For any symmetry, either a permutation $(\sigma_a\sigma_b\sigma_c)$ of (abc) appears in a row and then a permutation $(\sigma_c\sigma_f\sigma_i)$ of (cfi) appears in a column or *vice versa*. In the first case, the 9-j coefficient will be denoted by $(\sigma_a\sigma_b\sigma_c|\sigma_c\sigma_f\sigma_i)$ and, in the second case, by $(\sigma_c\sigma_f\sigma_i|\sigma_a\sigma_b\sigma_c)$. For example,

$$(bca|cif) = \begin{Bmatrix} b & c & a \\ a+d+i & i & a+d \\ e & f & d \end{Bmatrix} \qquad (ifc|acb) = \begin{Bmatrix} a+d & d & a \\ i & f & c \\ a+d+i & e & b \end{Bmatrix}. \tag{19}$$

This notation determines the 9-j coefficient uniquely and has the advantage that it is much shorter to tabulate than the 9-j symbol itself.

Among the 72 expressions, there are four single terms, namely

$$(abc|icf) \qquad (abc|ifc) \qquad (bac|cfi) \qquad \text{and} \qquad (bac|fci). \tag{20}$$

The first of these, in fact, corresponds to the second 9-j coefficient in (18).

Then, there are 16 symmetries for which the triple-sum expression reduces to a single sum and these are given in table 1.

Table 1. Symmetries giving rise to single-sum expressions.

Vandermonde ${}_2F_1$	$(cba\|cfi)$	$(cba\|fci)$	$(fic\|acb)$	$(cif\|acb)$	$(ifc\|cab)$	$(icf\|cab)$
Saalschütz ${}_3F_2$	$(bca\|cif)$	$(bca\|fic)$	$(fic\|bca)$	$(cif\|bca)$	$(icf\|acb)$	$(ifc\|acb)$
	$(cfi\|cab)$	$(fci\|cab)$				
Karlsson ${}_4F_3$	$(abc\|cfi)$	$(abc\|fci)$				

Of these 16 cases, six single sums are identified as a Vandermonde ${}_2F_1$ summation, eight are identified as a Saalschütz ${}_3F_2$ summation and two as a Karlsson ${}_4F_3$ summation. Thus, all the single sums can be explicitly performed using the theorems (5), (6) and (8). On the other hand, it also follows from the symmetries that all these sums can be written as a single term. In other words, suppose we did not know theorem (8), then we could have deduced it (in a restricted form only for integer parameters) from the equality $(abc|cfi) = (abc|icf)$. This inspired us to consider the other symmetries and to investigate whether the corresponding double or triple sums can be performed using known summation theorems or whether they give rise to new results.

Among the remaining symmetries, there are 20 double sums and 32 triple sums. The double sums fall into six different types, which we have denoted by D1, ..., D6. Among

Table 2. Symmetries giving rise to double-sum expressions.

D1	$(fic\|cba)$	$(cif\|cba)$					New
D2	$(bac\|fic)$	$(bac\|cif)cr$					New
D3	$(bca\|cfi)$	$(bca\|fci)$	$(icf\|bac)$	$(ifc\|bac)$			(7)+(5)
D4	$(acb\|cfi)$	$(acb\|fci)$	$(cba\|icf)$	$(cba\|ifc)$	$(fic\|cab)$	$(cif\|cab)$	(5)+(5)
D5	$(cab\|cfi)$	$(cab\|fci)$	$(fic\|abc)$	$(cif\|abc)$			(7)+(6)
D6	$(cfi\|acb)$	$(fci\|acb)$					(5)+(6)

Table 3. Symmetries giving rise to triple-sum expressions.

	$(abc\|fic)$	$(abc\|cif)$	$(bca\|icf)$	$(bca\|ifc)$	$(acb\|fic)$	$(acb\|cif)$	
T1	$(cab\|icf)$	$(cab\|ifc)$	$(bac\|icf)$	$(bac\|ifc)$	$(cfi\|abc)$	$(fci\|abc)$	New
	$(cfi\|bca)$	$(fci\|bca)$	$(cfi\|cba)$	$(fci\|cba)$	$(fic\|bac)$	$(cif\|bac)$	
T2	$(acb\|icf)$	$(acb\|ifc)$					New
T3	$(cab\|fic)$	$(cab\|cif)$	$(icf\|cba)$	$(ifc\|cba)$	$(icf\|bca)$	$(ifc\|bca)$	(7)+D1
T4	$(cba\|fic)$	$(cba\|cif)$	$(icf\|abc)$	$(ifc\|abc)$	$(cfi\|bac)$	$(fci\|bac)$	(5)+(6)+(6)

the 32 triple sums, only four different types appear, denoted by T1, T2, T3 and T4. Tables 2 and 3 give the double and triple sums according to their type.

It remains to describe these ten types D1, ..., D6, T1, ..., T4. A description is summarized in the last columns of tables 2 and 3. D1 and D2 seem to be new and they will be given at the end of this section. D3 is a double sum with one coupling, but nevertheless one of the summations can be performed separately by means of the Minton ${}_4F_3$ theorem (7). What remains can then be simplified and reduced to Vandermonde's theorem in the second summation variable. D4 is a double sum with one coupling, where one of the summations can be performed first by means of Vandermonde's ${}_2F_1$ theorem (5), and the remaining sum also reduces to a Vandermonde sum. Type D5 is a double sum, again with one coupling. Here, one of the summations can be done using Minton's ${}_4F_3$ theorem, and what remains is a summation of the Saalschütz ${}_3F_2$ type. Finally, for the D6 sums, one summation can be performed by means of Vandermonde's theorem and the remaining summation then reduces to a Saalschütz ${}_3F_2$ series.

For the triple sums, T1 and T2 seem to be new and will be given explicitly below. In the T3 triple sums, one summation can be performed by means of Minton's ${}_4F_3$ theorem, and the remaining double sum is of type D1. In the T4 triple sums, one summation can be performed using Vandermonde's theorem, then a second summation can be performed using the Saalschütz ${}_3F_2$ theorem and, finally, the third summation can also be done using the Saalschütz theorem.

To conclude this section, we give the explicit forms of the summations D1, D2, T1 and T2, which cannot be performed using any known summation theorems.

The double sum D1 is of the following type:

$$\sum_{x,z} \frac{(-1)^{x+z}}{x!z!} \frac{(2b-x)!(a-b+d+e+i+x)!(a-b+c+x)!}{(a+b+d-e+i-x)!(-a+b+c-x)!(a-b+d+e-i+x)!}$$
$$\times \frac{(2f-z)!(c-f+i+z)!}{(f+d+e+1-z)!(c+f-i-z)!(a-b-f+i+x+z)!}$$
$$= (-1)^{2c} \frac{(a-b+d+e+i)!(a+b+d+e+i+1)!(2i)!}{(e+d-f)!(2a+2d+2i+1)!(e+d+f+1)!}. \tag{21}$$

In this sum, the coupling comes from the factor $(a-b-f+i+x+z)$. But $a-b-f+i$ is non-positive; indeed, if it was positive then also $(a-b-f+i)+(d-e+f)>0$ or

$(a+d+i)-b-e>0$, which contradicts the fact that $(b,e,a+d+i)$ forms a triad in (17). Because of this, summation (21) does not have a term with $x=z=0$ as a starting value and it cannot directly be rewritten in terms of one of the double hypergeometric functions of the previous section. However, we shall see in the next section that by a simple substitution of the summation variables, (21) can be cast in the form of a double hypergeometric function.

The double sum D2 is of the type:

$$\sum_{y,z}\frac{(-1)^{y+z}}{y!z!}\frac{(e+d-f+y)!}{(2d+1+y)!(2a-y)!(e-d+f-y)!}\times$$
$$\frac{(2b-z)!(a-b+c+z)!(a+b+i+f-y-z)!}{(a+b+d+e+i+1-z)!(a+b+d-e+i-z)!(-a+b+c-z)!(a-b+f-i+z)!}$$
$$=(-1)^{c+d-e+i}\frac{(2d)!}{(d+e+f+1)!(2a+2d+1)!(d-e+f)!}. \tag{22}$$

Here, there is again a coupling, but now we can assume that $2a$ determines the termination of the y summation and that $-a+b+c$ determines the termination of the z summation. Then, the coupling itself never becomes zero and expression (22) can indeed be written in terms of a double hypergeometric function provided that $a-b+f-i\geqslant 0$. This will be given explicitly in the next section.

The triple sum T1 can be written in the following form:

$$\sum_{x,y,z}\frac{(-1)^{x+y+z}(2a+2d+x)!(c+f-i+x)!(a+b+d-e+i+y)!(a+b-c+y)!}{x!y!z!(-c+f+i-x)!(2b+1+y)!(a-b+d+e+i-y)!(a-b+c-y)!}$$
$$\times\frac{(-d+e+f+z)!}{(2a+2d+1-z)!(d-e+f-z)!}\frac{(p_1-y-z)!}{(p_2+x+y)!(p_3+x+z)!}$$
$$=\frac{(-1)^{c+d-e-i}(2a+2d)!(a+b+d-e+i)!(-d+e+f)!}{(-c+f+i)!(a-b+c)!(a+b+c+1)!(b+e-a-d-i)!(d-e+f)!} \tag{23}$$

where $p_1=a-b+c+2d$, $p_2=a+d+b-e-i$ and $p_3=-d+e-i+c$. The upper bounds for the summation variables x,y,z are given by $-c+f+i, a-b+c, d-e+f$ respectively. Thus p_1-y-z is always positive, and provided that p_2 and p_3 are also positive (which is possible in this case), the summation (23) can be rewritten as a genuine triple hypergeometric function; this is presented in the next section.

Finally, the triple sum T2 is of the type:

$$\sum_{x,y,z}\frac{(-1)^{x+y+z}}{x!y!z!}\frac{(2b-x)!(a-b+c+x)!(a-b+d+e+i+x)!}{(-a+b+c-x)!(a+b+d-e+i-x)!}$$
$$\times\frac{(c+f-i+y)!(d-e+f+y)!(2i+z)!}{(2f+1+y)!(c-f+i-y)!(d+e-f-y)!(2a+2d+1-z)!(2a-z)!}$$
$$\times\frac{(p_1-y-z)}{(p_2+x+y)!(p_3+x+z)!}$$
$$=\frac{(-1)^{2a+c+d-e+i}(a-b+d+e+i)!(c+f-i)!(2i)!(2d)!(a+b+d+e+i+1)!}{(2a+2d+1)!(c-f+i)!(d+e-f)!(c+f+i+1)!(d+e+f+1)!(-c+f+i)!} \tag{24}$$

where $p_1 = 2a + d + e - f$, $p_2 = a - b + f - i$ and $p_3 = -a - b - d + e + i$. Again, under certain extra assumptions, this summation can be rewritten as a triple hypergeometric function, given in the following section.

Once more, it should be emphasized that the summations (21)–(24) cannot be simplified using known summation theorems. In addition, when writing, for example, the summation explicitly for some of the other T1 symmetries (as given in table 3), the actual expression might, at first sight, look different from (23). Only when rewritten in the multiple hypergeometric function notation do such apparently different summations clearly become identical.

4. New summation formulae

Since (21) is somewhat special, we shall first treat the hypergeometric function summation that is related to (22).

Expression (22) is valid for all integers or half-integers a, b, c, d, e, f, i that satisfy the triangular relations implied by (17). To write it in a more general form, it is useful to relabel these seven independent parameters by the following seven integer parameters:

$$\begin{aligned} m &= a + b - c \\ p &= d - e + f \\ q &= c - f + i \\ r &= a - b + f - i \end{aligned} \qquad \begin{aligned} \alpha &= -a - b - d - e - i - 1 \\ \beta &= -2b \\ \gamma &= d + e - f + 1. \end{aligned} \tag{25}$$

Then, (22) can be rewritten in terms of the double hypergeometric function (10):

$$\begin{aligned} F^{0:4;3}_{1:2;1}\left[\begin{matrix} - \\ \alpha+\gamma \end{matrix} : \begin{matrix} \alpha, \beta+m, q+r+1, -m-p-q \\ \beta, r+1 \end{matrix} ; \right. \\ \left. \begin{matrix} \alpha+\gamma+m+p+q, \gamma, -m-q-r \\ \gamma+p+1 \end{matrix} ; 1, 1 \right] \\ = (-1)^{m+q} \frac{(\gamma)_p (\alpha)_{m+q}}{(\beta)_m (\alpha+\gamma)_{m+p+q} (\gamma+p+1)_{m+q+r}} \frac{(m+q+r)!(m+p+q)!r!}{p!(q+r)!}. \end{aligned} \tag{26}$$

In this form, it turns out that the above expression is valid for all complex numbers α, β, γ and for all non-negative integers m, p, q, r, as long as the termination of the series is determined by $(-m - p - q)$ for the first summation index and by $(-m - q - r)$ for the second summation index. Thus, (26) is a summation theorem for a special double hypergeometric function. The validity of (26) has been checked for a large number of data by means of MACSYMA (1985). In principle, our technique provides a proof of (26) only for integer values (due to integer or half-integer angular momenta). However, the more general result (26) can be understood by extending some angular momentum values to the complex plane and by considering analytic continuations of angular momentum coefficients (thus, certain couplings correspond to $su(1, 1)$ tensor products rather than to $su(2)$ tensor products). The property observed here is similar to the extension of some properties of the 6-j coefficients to certain complex arguments (Raynal 1979). It should be mentioned that a detailed study of the definition and properties of generalized 9-j coefficients has not appeared in the literature.

Some summations for special Kampé de Fériet functions (9) can be obtained from (26) by putting $m=0$ or $q=0$. For example, when $q=0$, (26) becomes

$$F^{0:3}_{1:1}\left[\begin{matrix} - \\ \alpha+\gamma \end{matrix} : \begin{matrix} \alpha,\beta+m,-m-p \\ \beta \end{matrix} ; \begin{matrix} \alpha+\gamma+m+p,\gamma,-m-r \\ \gamma+p+1 \end{matrix} ; 1,1\right]$$
$$= (-1)^m \frac{(\gamma)_p(\alpha)_m}{(\beta)_m(\alpha+\gamma)_{m+p}(\gamma+p+1)_{m+r}} \frac{(m+r)!(m+p)!}{p!}. \tag{27}$$

Now consider the double sum (21). As was explained in the previous section, there is no term with $x=z=0$, which prevents us from rewriting (21) directly as a hypergeometric function. However, if we perform the following transformation of variables:

$$x=-a+b+c-u \qquad z=c+f-i-v$$

then (21) becomes

$$\sum_{u,v} \frac{(-1)^{u+v}}{u!v!} \frac{(a+b-c+u)!(c+d+e+i-u)!(2c-u)!}{(2a-c+d-e+i+u)!(-a+b+c-u)!(c+d+e-i-u)!}$$
$$\times \frac{(-c+f+i+v)!(2c-v)!}{(-c+d+e+i+1+v)!(c+f-i-v)!(2c-u-v)!}$$
$$= (-1)^{a-b-f+i} \frac{(a-b+d+e+i)!(a+b+d+e+i+1)!(2i)!}{(e+d-f)!(2a+2d+2i+1)!(e+d+f+1)!}. \tag{28}$$

Now there is a term with $u=v=0$ and one can again rewrite the equation. It is convenient to define the following parameters:

$$\begin{aligned} n_0 &= a-b+c \\ n_1 &= -a+b-d+e-i \\ n_2 &= d-e+f \\ n_3 &= c-f+i \end{aligned} \qquad \begin{aligned} \alpha &= -c+d+e+i+1 \\ \beta &= -c+f+i+1 \\ \gamma &= 2a-c+d-e+i+1 \end{aligned} \tag{29}$$

and to set $n=n_0+n_1+n_2+n_3$. Then, (28) can be rewritten in terms of the double hypergeometric function (10):

$$F^{1:3;2}_{0:3;2}\left[\begin{matrix} -n \\ - \end{matrix} : \begin{matrix} \gamma+n_1,\beta-\alpha-n+n_3,-n+n_0 \\ \gamma,1-\alpha-n,-n \end{matrix} ; \begin{matrix} \beta,-n+n_3 \\ \alpha+1,-n \end{matrix} ; 1,1\right]$$
$$= (-1)^{n_1+n_2} \frac{(n-n_0)!(n-n_3)!(\alpha-\beta+1)_{n-n_3}(\beta)_{n_3}(\alpha+\gamma+n)_{n_1}}{n!(\alpha+n_0)_{n-n_0}(\alpha+1)_{n-n_3}(\gamma)_{n_1}}. \tag{30}$$

In this form, the summation is valid for all non-negative integers n_0, n_1, n_2 and n_3, and for all complex numbers α, β, γ (as long as no denominators become zero). Some summation theorems for Kampé de Fériet functions of type $F^{1:2}_{0:2}$ could be derived from (30), for example by putting $\beta = 1-n_3$ or by considering the special cases $n_0=0$ or $n_1=0$.

Next, we consider (23), and use the following relabelling:

$$\begin{aligned} \alpha &= -a+b-2d-f-i & p &= -c+f+i \\ \beta &= -a+b-d+e-i+1 & q &= a-b+c \\ \gamma &= a+b-i-f+1 & r &= d-e+f. \\ \delta &= 2b+2 \end{aligned} \tag{31}$$

With these new parameters, the triple sum (23) can be rewritten as a triple hypergeometric function (11) as follows.

$$F^{0:3;4;3}_{3:0;1;0}\left[\begin{matrix} - \\ \alpha+p:011,\beta+q:101,\gamma+r:110 \end{matrix}:\begin{matrix}\gamma-\alpha,\beta+q+r,-p \\ - \end{matrix};\begin{matrix}\delta-\beta,\gamma+p,\alpha+r,-q \\ \delta\end{matrix};\begin{matrix}\beta+p+q,\alpha-\gamma,-r \\ -\end{matrix};1,1,1\right] = \frac{(\alpha)_p(\beta)_q(\gamma)_r}{(\gamma)_p(\delta)_q(\alpha)_r}. \tag{32}$$

In this form, the above summation result is valid for all complex parameters $\alpha, \beta, \gamma, \delta$ and for all non-negative integers p, q, r (which determine the termination of the three summation indices). As for the previous case, it has been verified carefully using MACSYMA.

An interesting symmetric double summation follows from (32) by putting $q = 0$. Then, (32) reduces to a Kampé de Fériet function and we obtain

$$F^{0:3}_{1:1}\left[\begin{matrix} - \\ \beta\end{matrix}:\begin{matrix}\gamma-\alpha,\beta+r,-p \\ \gamma+r\end{matrix};\begin{matrix}\alpha-\gamma,\beta+p,-r \\ \alpha+p\end{matrix};1,1\right] = \frac{(\alpha)_p(\gamma)_r}{(\gamma)_p(\alpha)_r}. \tag{33}$$

This terminating series summation can, in fact, be further generalized to

$$F^{0:3}_{1:1}\left[\begin{matrix} - \\ b\end{matrix}:\begin{matrix}c-a,b+e,-d \\ c+e\end{matrix};\begin{matrix}a-c,b+d,-e \\ a+d\end{matrix};1,1\right] = \frac{\Gamma(a+d)\Gamma(c+e)}{\Gamma(c+d)\Gamma(a+e)} \tag{34}$$

which is now valid for all complex parameters for which the double series is convergent (that is, for which $\mathrm{Re}(a+d) > 0$ and $\mathrm{Re}(c+e) > 0$). We have performed a number of numerical tests to verify (34).

Finally, we give the triple hypergeometric function result related to (24). The new parameters are now

$$\begin{aligned} & & m &= c-f+i \\ \alpha &= -a+b+c & p &= a-b+f-i \\ \beta &= 2f+1 & q &= a+b-c \\ \gamma &= -f+e+d & r &= d-e+f. \end{aligned} \tag{35}$$

Then, (24) can be rewritten as follows.

$$\begin{aligned} F^{0:4;4;3}_{3:1;1;0}&\left[\begin{matrix} - \\ 1-\gamma-m-p-q:011,\beta-\alpha-p-q-r:101,p+1:110\end{matrix}:\right. \\ &\begin{matrix}m+p+1,m+\gamma+\beta-\alpha-1,1-\alpha,-m-q-r \\ 1-\alpha-q\end{matrix};\begin{matrix}\alpha+p,r+1,1-\gamma,-m \\ \beta\end{matrix}; \\ &\left.\begin{matrix}\beta-\alpha+m-p,-\gamma-m-p-q-r,-m-p-q \\ -\end{matrix};1,1,1\right] \\ &= \frac{(-1)^{p+q+r}(m+q+r)!(m+p+q)!p!(\beta+\gamma-1)_{m+q}(\gamma)_r}{(m+p)!r!(\alpha)_q(\beta)_m(\gamma)_{m+p+q}(\beta-\alpha-p-q-r)_{q+r}}. \end{aligned} \tag{36}$$

Again, this summation result is valid for all complex α, β, γ and all non-negative integers m, p, q, r such that the termination of the three summation indices is determined by $-m-q-r$, $-m$ and $-m-p-q$ respectively.

To conclude, by extending certain arguments, originally corresponding to angular momenta, to the complex plane, we have been able to formulate four summation theorems, (26), (30), (32) and (36). These summations have seven independent parameters, some of which are non-negative integers. By specializing certain parameters in these expressions, simple summation theorems for some Kampé de Fériet functions can be obtained, such as (27), (33) and (34).

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Appendix

The summation results were derived from the central formulae (21)–(24). Each of these can still be manipulated to give interesting side results. We give one example in this appendix.

Consider (22) as an initial expression, and consider (22) with c replaced by $c+1$ as a second expression. Adding the right-hand sides of these two expressions obviously gives zero and since c appears in only two factors in the left-hand sides the sum of these also simplifies greatly. Thus, we obtain

$$\sum_{y,z} \frac{(-1)^{y+z}}{y!z!} \frac{(e+d-f+y)!}{(2d+1+y)!(2a-y)!(e-d+f-y)!}$$
$$\times \frac{(2b-z)!(a+b+i+f-y-z)!}{(a+b+d+e+i+1-z)!(a+b+d-e+i-z)!(a-b+f-i+z)!}$$
$$\times \frac{(a-b+c+z)!}{(-a+b+c+1-z)!} = 0.$$

Using the same relabelling as in (25), this can be rewritten as

$$F^{0:4;3}_{1:2;1}\left[\begin{matrix} - \\ \alpha+\gamma \end{matrix} : \begin{matrix} \alpha, \beta+m-1, q+r+1, -m-p-q \\ \beta, r+1 \end{matrix} ; \right.$$
$$\left. \begin{matrix} \alpha+\gamma+m+p+q, \gamma, -m-q-r \\ \gamma+p+1 \end{matrix} ; 1, 1 \right] = 0.$$

Again, this is valid for all non-negative integers m, n, p, q and all complex α, β, γ (as long as the denominators are non-zero).

Another remark which is worth pointing out is that some of the summation theorems given here have a q-analogue. This leads to new results for double *basic* hypergeometric function sums (Gasper and Rahman 1990). For this purpose, we define the following double basic hypergeometric function, which is a q-generalization of a Kampé de Fériet function:

$$\Phi^{0:3}_{1:1}\left[\begin{matrix} - \\ c \end{matrix} : \begin{matrix} a_1, a_2, a_3 \\ a_4 \end{matrix} ; \begin{matrix} b_1, b_2, b_3 \\ b_4 \end{matrix} ; x, y; q\right] = \sum_{m,n=0}^{\infty} \frac{(a_1, a_2, a_3; q)_m}{(a_4; q)_m} \frac{(b_1, b_2, b_3; q)_n}{(b_4; q)_n} \frac{1}{(c; q)_{m+n}} \frac{x^m}{(q; q)_m} \frac{y^n}{(q; q)_n} \tag{37}$$

where the classical notation of Gasper and Rahman (1990) is used:

$$(a; q)_n = \begin{cases} 1 & n = 0 \\ (1-a)(1-aq)\cdots(1-aq^{n-1}) & n = 1, 2, \ldots \\ \prod_{k=0}^{\infty}(1-aq^k) \qquad (|q| < 1) & n = \infty \end{cases} \tag{38}$$

and $(a_1, a_2, \ldots, a_m; q) = (a_1; q)_n(a_2; q)_n \cdots (a_m; q)_n$. Then, the following q-analogues of (33) and (34) are valid:

$$\Phi^{0:3}_{1:1}\left[\begin{matrix} - \\ \beta \end{matrix} : \begin{matrix} \gamma/\alpha, \beta q^r, q^{-p} \\ \gamma q^r \end{matrix} ; \begin{matrix} \alpha/\gamma, \beta q^p, q^{-r} \\ \alpha q^p \end{matrix} ; q, q; q\right] = (\gamma/\alpha)^{p-r} \frac{(\alpha; q)_p (\gamma; q)_r}{(\gamma; q)_p (\alpha; q)_r} \tag{39}$$

where p and r are positive integers; and

$$\Phi^{0:3}_{1:1}\left[\begin{matrix} - \\ \beta \end{matrix} : \begin{matrix} \gamma/\alpha, \beta\epsilon, 1/\delta \\ \gamma\epsilon \end{matrix} ; \begin{matrix} \alpha/\gamma, \beta\delta, 1/\epsilon \\ \alpha\delta \end{matrix} ; q, q; q\right] = \frac{(\frac{1}{\gamma\delta}, \frac{1}{\alpha\epsilon}; q^{-1})_\infty}{(\frac{1}{\alpha\delta}, \frac{1}{\gamma\epsilon}; q^{-1})_\infty} \tag{40}$$

where $|q|, |\alpha\delta|, |\gamma\epsilon| > 1$.

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